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A Strong Maximum Principle for parabolic equations with the p-Laplacian ☆

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ABSTRACT

We prove the Strong Maximum Principle (SMP) under suitable assumptions for a class of quasilinear parabolic problems with the p-Laplacian, $p > 1$, on bounded cylindrical domains of $\mathbb{R}^{N+1}$,

$$\partial_t u - \Delta_p u - \lambda |u|^{p-2} u \geq 0,$$

with nonnegative initial–boundary conditions and $\lambda \leq 0$, and we give some counterexamples to the SMP if some of our assumptions are violated. We show that the Hopf Maximum Principle holds for $1 < p < 2$, and give a counterexample to it for $p > 2$. Also the Weak Maximum Principle for $\lambda \leq \lambda_1$ is established.

1. Introduction

Throughout this article, $\Omega_T := \Omega \times (0, T)$ denotes an $(N + 1)$-dimensional cylinder, where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with the boundary $\partial\Omega$, $T \in (0, +\infty)$, and $\partial\Omega_T := \partial\Omega \times (0, T)$ denotes its cylindrical surface. The boundary $\partial\Omega$ is assumed to be a compact manifold of class at least C^1. We consider the following parabolic problem:

$$\left.\begin{aligned} &\mathcal{L}_\lambda[u] := \partial_t u - \Delta_p u - \lambda |u|^{p-2} u \geq 0, && (x,t) \in \Omega_T, \\ &u_0(x) := u(x,0) \geq 0, && x \in \Omega, \\ &u(x,t) \geq 0, && (x,t) \in \partial\Omega_T. \end{aligned}\right\} \tag{$\mathcal{P}$}$$

Here $\lambda \in \mathbb{R}$, $p > 1$, $\partial_t u := \partial u / \partial t$, and $\Delta_p u := \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u)$ is the p-Laplacian, with the spatial gradient $\nabla_x u$. Assume also that $u_0 \in W^{1,p}(\Omega)$, where $W^{1,p}(\Omega)$ is a standard Sobolev space.

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We deal with *weak supersolutions* (further referred to as *solutions*) of problem $(\mathcal{P})$ (see [4,10] for more details), i.e., with Lebesgue-measurable functions $u : \Omega_T \to \mathbb{R}$ satisfying

$$u \in C\big([0,T] \to L^2(\Omega)\big) \cap L^p\big((0,T) \to W^{1,p}(\Omega)\big)$$

and

$$\int\limits_{\Omega} u\varphi \, dx\Big|_0^\tau + \int\limits_{\Omega_\tau} \left(-u\frac{\partial \varphi}{\partial t} + |\nabla_x u|^{p-2}(\nabla_x u, \nabla_x \varphi)\right) dx\, dt - \lambda \int\limits_{\Omega_\tau} |u|^{p-2} u\varphi \, dx\, dt \ge 0$$

for every $\tau \in (0,T]$ and for all bounded nonnegative testing functions

$$\varphi \in W^{1,2}\big((0,\tau) \to L^2(\Omega)\big) \cap L^p\big((0,\tau) \to W_0^{1,p}(\Omega)\big) \quad \text{with } \varphi \ge 0 \text{ a.e. in } \Omega_\tau.$$

We remark that the *boundary trace* mapping $u \mapsto u|_{\partial\Omega}: W^{1,p}(\Omega) \to W^{1-(1/p),p}(\partial\Omega)$ ($\hookrightarrow L^p(\partial\Omega)$) is continuous (i.e., bounded) by J. Nečas [12, §2.5].

In the simplest and most common sense, the Strong Maximum Principle (abbreviated as SMP) says that any nontrivial solution of $(\mathcal{P})$ must be strictly positive in Ω_T, whereas the Weak Maximum Principle (WMP) asserts that the solution is just nonnegative in Ω_T. The Hopf Maximum Principle (HMP) complements the SMP and states that if the positive solution has an outer normal derivative at the boundary point of Ω_T, then this derivative must be negative.

In this article we show that, in general, the SMP and HMP cannot be satisfied without some restrictions on the initial data, or because of certain "intrinsic" properties of the p-Laplacian, such as degeneracy or singularity at points where the gradient vector field of the supersolution of $(\mathcal{P})$ vanishes.

Let us state our main results.

Theorem 1.1 *(SMP). Let $\lambda \le 0$, $u_0 \in W^{1,p}(\Omega)$, and assume that $u \in C^1(\Omega_T)$ satisfies $(\mathcal{P})$.*

1. *If $p > 2$ and $u_0 > 0$ in Ω, then $u > 0$ in Ω_T.*
2. *If $1 < p < 2$ and $u_0 \not\equiv 0$ in Ω, then there exists $\bar{t} \in (0,T]$ such that $u > 0$ in $\Omega_{\bar{t}}$.*

Note that we are not able to remove the assumption $u_0 > 0$ in Ω in the first case, nor are we able to guarantee the SMP on the whole space–time domain Ω_T in the second case. Section 4 contains the corresponding counterexamples.

For the sake of completeness, we give the SMP for $1 < p < 2$ on hyperplanes $\{(x,t) \in \Omega_T\colon\ t = t_0\}$, which has been proved in [11, Theorem 1, Part 1, p. 98] under additional assumptions on the regularity of u_0 and $u_0 \not\equiv 0$ in Ω.

Theorem 1.2. *Let $\lambda \le 0$, $1 < p < 2$ and assume that $u \in C^1(\Omega_T)$ satisfies $(\mathcal{P})$ with $u_0 \ge 0$ in Ω. If there exists $(x,t) \in \Omega_T$ such that $u(x,t) = 0$, then $u(\cdot,t) \equiv 0$ in Ω.*

The next clarification of maximum properties of solutions of $(\mathcal{P})$ is the Hopf Maximum Principle, given by the next theorem.

Theorem 1.3 *(HMP). Let Ω have a C^2-boundary $\partial\Omega$, $\lambda \le 0$, and $u_0 \in W^{1,p}(\Omega)$. Assume that $u \in C^1(\Omega_T \cup \{(x_1,t_1)\})$ satisfies $(\mathcal{P})$ and $u(x_1,t_1) = 0$ for some $(x_1,t_1) \in \partial\Omega_T$. If $1 < p < 2$, $u_0 \not\equiv 0$ in Ω, and the SMP holds for u in $\Omega_{\bar{t}}$ with $t_1 < \bar{t} \le T$, then the outer normal derivative at (x_1,t_1) is negative, i.e.*

$$\frac{\partial u(x_1,t_1)}{\partial \nu} < 0,$$

where ν is the outer unit normal to $\partial\Omega_T$ at (x_1,t_1).

In Section 4 we also present a counterexample to the HMP for the case $p > 2$.

The proofs of theorems above mostly take advantage of the Weak Comparison Principle (WCP), which says that if for some u, v we have $\mathcal{L}_\lambda[u] \le \mathcal{L}_\lambda[v]$ in the domain E and $u \le v$ on the boundary ∂E, then $u \le v$ in E. The proof of the WCP is known under slightly stronger conditions (see, e.g., [13]), but it can be generalized directly to our case. However, we show it in Section 2 for completeness.

In addition to the WCP, there arises the question about the validity of the Strong Comparison Principle (SCP), which states that if $\mathcal{L}_\lambda[u] \le \mathcal{L}_\lambda[v]$ in E and $u \le v$ on ∂E, then either $u \equiv v$, or $u < v$ in E. In Section 4 we make some remarks in this direction.

For a better understanding of the problem notice that if u satisfies $(\mathcal{P})$ and $\partial_t u \le 0$ on $\Omega_{\bar t}$ for some $\bar t \in (0, T]$, then the SMP and HMP hold on $\Omega_{\bar t}$ for $\lambda \le 0$. Moreover, if we assume in addition that $u = 0$ on $\partial\Omega_{\bar t}$, then the SMP and HMP hold on $\Omega_{\bar t}$ for $\lambda < \lambda_1$ (see [2,3]). Indeed, by transferring $\partial_t u$ to the right-hand side of $(\mathcal{P})$, we arrive at

$$-\Delta_p u - \lambda|u|^{p-2}u \ge 0, \quad (x,t) \in \Omega \times \{t_0\},$$

for all $t_0 \in (0, \bar t\,]$, and hence get the SMP and HMP. Hereinafter, we denote by λ_1 the first eigenvalue of the negative Dirichlet p-Laplacian $-\Delta_p$ in Ω:

$$\lambda_1 \stackrel{\text{def}}{=} \left\{ \int_\Omega |\nabla u|^p \, dx\colon\ u \in W_0^{1,p}(\Omega) \text{ with } \int_\Omega |u|^p \, dx = 1 \right\}.$$

It is known that $\lambda_1 > 0$ and λ_1 is simple due to the result of Anane [1, Théorème 1, p. 727].

Maximum and comparison principles are among the essential tools in the theory of linear and nonlinear PDE's. The main areas of their application are proving existence or nonexistence and uniqueness or multiplicity of solutions to boundary value problems. Among many other references, the classical ones for linear PDEs are Friedman [8] and Protter & Weinberger [14].

Maximum principles for nonlinear *elliptic* PDEs have been studied intensively by Pucci & Serrin [15], Vázquez [16], and many others. A significant progress in studying the comparison principles for such kind of PDEs has been made in Cuesta & Takáč [2,3]. At the same time, to our best knowledge, there are very few articles concerning the maximum and comparison principles for nonlinear (especially *quasilinear*) *parabolic* problems. We can mention only the works of Nazaret [11] and Vétois [17]. The first article treats the SMP for weak solutions of $(\mathcal{P})$ with $\lambda \le 0$ on hyperplanes $\{(x,t) \in \Omega_T\colon\ t = t_0\}$. The proof of [11, Theorem 1, Part 2, p. 98] for $p > 2$ needs some additional, rather strong assumptions on the initial function u_0, such as $u_0 > 0$ in $\overline{\Omega}$ or $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, since in these cases one can guarantee the strict positivity of t_0 defined in [11, p. 99],

$$t_0 \stackrel{\text{def}}{=} \inf\bigl\{t > 0\colon\ \exists x \in \Omega,\ u(x,t) = 0\bigr\}.$$

Indeed, otherwise there might exist a sequence $(x_n, t_n) \in \Omega \times (0, T)$ such that $u(x_n, t_n) = 0$ with $x_n \to x_0 \in \partial\Omega$ and $t_n \to t_0 = 0$ as $n \to +\infty$. Nevertheless, our Theorem 1.1, Part 1, shows that for $p > 2$ and $u_0 > 0$ in Ω the solution is strictly positive in the whole of Ω_T. Paper [17] deals with the anisotropic parabolic p-Laplacian equations in a setting different from ours. The special case of equation $\partial_t u - \Delta_p u = 0$ in Ω_T is treated in DiBenedetto [4], Chapter VI (for $2 < p < \infty$) and Chapter VII (for $\frac{2N}{N+1} < p < 2$), where a kind of local strong maximum principle is obtained from local Harnack's inequality. Further results on a local strong maximum principle and local Harnack's inequality for this equation, including some counterexamples, can be found in the recent monograph by DiBenedetto, Gianazza, and Vespri [7] and in their articles [5,6]. The aim of the present paper is to examine more carefully the maximum and comparison principles for $(\mathcal{P})$.

The article is organized as follows. In Section 2 we prove the WCP for $\lambda \le 0$ and the WMP for $\lambda \le \lambda_1$. Section 3 contains the proofs of the main results – Theorem 1.1 and Theorem 1.3. Finally, in Section 4 we present some counterexamples.

2. Weak maximum and comparison principles

In this section we prove the WCP in Theorem 2.1 and the WMP in Theorem 2.4. Using Theorem 2.1 we prove an auxiliary result, Lemma 2.3, which will be used in the proof of the SMP (Theorem 1.1) in Section 3.

Theorem 2.1 *(WCP). Let $\lambda \le 0$, $u_0, v_0 \in W^{1,p}(\Omega)$, and let*

$$u, v \in C\big([0,T] \to L^2(\Omega)\big) \cap L^p\big((0,T) \to W^{1,p}(\Omega)\big) \tag{2.1}$$

with $\partial_t u, \partial_t v \in L^2(\Omega_T)$ satisfy

$$\left.\begin{array}{ll} \mathcal{L}_\lambda[u] \le \mathcal{L}_\lambda[v], & (x,t) \in \Omega_T, \\ u_0 \le v_0, & x \in \Omega, \\ u \le v, & (x,t) \in \partial\Omega_T. \end{array}\right\} \tag{2.2}$$

Then $u \le v$ in Ω_T.

Proof. 1. Let $(u-v)^+ := \max\{u-v, 0\}$ and assume, by contradiction, that

$$\Omega_T^+ := \big\{(x,t) \in \Omega_T\colon\ u(x,t) > v(x,t)\big\}$$

has positive Lebesgue measure in $\mathbb{R}^{N+1}$. Obviously $(u-v)^+ \ge 0$ in Ω_T and

$$(u-v)^+ \in C\big([0,T] \to L^2(\Omega)\big) \cap L^p\big((0,T) \to W^{1,p}(\Omega)\big),$$

by assumption (2.1). Testing the first inequality of (2.2) by $(u-v)^+$ we get

$$\begin{aligned} &\int\limits_{\Omega_T^+} \frac{\partial(u-v)}{\partial t}(u-v)\,dx\,dt + \int\limits_{\Omega_T^+} \big(|\nabla_x u|^{p-2}\nabla_x u - |\nabla_x v|^{p-2}\nabla_x v\big)(\nabla_x u - \nabla_x v)\,dx\,dt \\ &\quad - \lambda \int\limits_{\Omega_T^+} \big(|u|^{p-2}u - |v|^{p-2}v\big)(u-v)\,dx\,dt \le 0. \end{aligned} \tag{2.3}$$

Note that by Hölder's inequality, (2.1), and $\partial_t u, \partial_t v \in L^2(\Omega_T)$, we obtain

$$\int\limits_{\Omega_T^+} \left|\frac{\partial(u-v)}{\partial t}\right| |u-v|\,dx\,dt \le \big\|\partial_t(u-v)\big\|_{L^2(\Omega_T)} \cdot \big\|(u-v)\big\|_{L^2(\Omega_T)} < +\infty;$$

consequently, the first term in (2.3) exists as a Lebesgue integral. Using this fact we conclude that

$$\begin{aligned} \int\limits_{\Omega_T^+} \frac{\partial(u-v)}{\partial t}(u-v)\,dx\,dt &\equiv \frac{1}{2}\int\limits_{\Omega}\int\limits_0^T \frac{\partial}{\partial t}\big((u-v)^+\big)^2\,dt\,dx \\ &= \frac{1}{2}\left[\int\limits_\Omega \big((u-v)^+(x,T)\big)^2\,dx - \int\limits_\Omega \big((u-v)^+(x,0)\big)^2\,dx\right] \ge 0, \end{aligned} \tag{2.4}$$

where the nonnegativity follows from the assumption $u_0 \le v_0$, hence, $(u-v)^+(x,0) \equiv 0$.

On the other hand, using convexity of $u \in W_0^{1,p} \mapsto \frac{1}{p}\int_\Omega |\nabla u|^p dx$ we deduce

$$\int_{\Omega_T^+} \big(|\nabla_x u|^{p-2}\nabla_x u - |\nabla_x v|^{p-2}\nabla_x v\big)(\nabla_x u - \nabla_x v)\, dx\, dt \geq 0,$$

for all $p > 1$; and, in the same way, the third term in (2.3) is also nonnegative, since $\lambda \leq 0$. Thus, (2.3) is possible if and only if every term is equal to zero. By our assumption $u \leq v$ on $\partial\Omega_T$ and (2.4), we get $(u - v)^+ = 0$ on $\partial\Omega_T^+$. Since $\nabla_x u = \nabla_x v$ in Ω_T^+ we conclude that $u \equiv v$ in Ω_T^+, which contradicts our assumption that $u > v$ on a set of positive Lebesgue measure in $\mathbb{R}^{N+1}$. □

Remark 2.2. The Weak Maximum Principle (WMP) for $\lambda \leq 0$ follows directly from Theorem 2.1, by taking $u \equiv 0$ in $\overline{\Omega}_T$.

Later, in order to prove the SMP we need another version of the WCP in more general space–time domains, but with restrictions on boundary conditions. We impose rather strong regularity hypotheses (C^1) that enable a simple, direct application of this lemma to our needs.

Lemma 2.3. *Let $E \subset \mathbb{R}^{N+1}$ be a bounded domain with boundary ∂E. Assume $\lambda \leq 0$ and $u, v \in C^1(\overline{E})$ satisfy*

$$\begin{cases} \mathcal{L}_\lambda[u] \leq \mathcal{L}_\lambda[v], & (x,t) \in E, \\ u \leq v, & (x,t) \in \partial E. \end{cases}$$

Then $u \leq v$ in E.

Proof. Since E is bounded, there exists a cylinder $B \times (t_1, t_2) \subset \mathbb{R}^{N+1}$ with a bounded base $B \subset \mathbb{R}^N$ such that $E \subset B \times (t_1, t_2)$. Arguing now as in the proof of Theorem 2.1 we get a contradiction. Thus, $u \leq v$ in E. □

Now we prove the WMP for $\lambda \leq \lambda_1$ in the following formulation.

Theorem 2.4. *Let $\lambda \leq \lambda_1$ and assume that u satisfies weakly*

$$\begin{cases} \mathcal{L}_\lambda[u] \geq 0, & (x,t) \in \Omega_T, \\ u_0 \geq 0, & x \in \Omega, \\ u = 0, & (x,t) \in \partial\Omega_T, \end{cases}$$

with $u \in L^2(\Omega_T)$. Then $u \geq 0$ in Ω_T.

Proof. Note that for $\lambda \leq 0$ the WMP follows from Theorem 2.1. Therefore, we will assume $\lambda \in (0, \lambda_1]$.

Denote $u^- \stackrel{\text{def}}{=} \max\{-u, 0\}$ and suppose, contrary to our claim, that

$$\Omega_T^- := \big\{(x,t) \in \Omega_T\colon\ u(x,t) < 0\big\} \neq \emptyset.$$

Testing $\mathcal{L}_\lambda[u] \geq 0$ by u^- we get

$$\int_{\Omega_T^-} \frac{\partial u}{\partial t} u\, dx\, dt + \int_{\Omega_T^-} |\nabla_x u|^p\, dx\, dt - \lambda \int_{\Omega_T^-} |u|^p\, dx\, dt \leq 0.$$

On one hand, similarly to (2.4) we find that the first term is nonnegative. On the other hand, since $\Omega_T^- \subseteq \Omega_T$, we get

$$\int\limits_{\Omega_T} \left|\nabla_x (u^-)\right|^p dx\, dt - \lambda \int\limits_{\Omega_T} \left|u^-\right|^p dx\, dt \geq (\lambda_1 - \lambda) \int\limits_{\Omega_T} \left|u^-\right|^p dx\, dt \geq 0,$$

for $\lambda \leq \lambda_1$. Hence, arguing as in the proof of Theorem 2.1 we conclude that $u^- = 0$ in Ω_T and, consequently, obtain a contradiction. $\square$

3. Strong Maximum Principle

In this section we prove the SMP for $(\mathcal{P})$. Hereinafter, we denote by

$$B_r(\bar{x}, \bar{t}\,) := \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R}:\ d(x,t;\bar{x},\bar{t}\,) < r\right\}$$

the open ball of radius $r > 0$ centered at a point $(\bar{x}, \bar{t}\,)$ in $\mathbb{R}^{N+1}$. Here

$$d(x,t;\bar{x},\bar{t}\,) = \sqrt{|x-\bar{x}|^2 + (t-\bar{t}\,)^2} := \sqrt{\sum_{i=1}^{N} (x_i - \bar{x}_i)^2 + (t-\bar{t}\,)^2} \tag{3.1}$$

is the standard Euclidean distance in $\mathbb{R}^{N+1}$ between (x,t) and $(\bar{x}, \bar{t}\,)$.

First of all, we need the following auxiliary lemma.

Lemma 3.1. *Let $\lambda \in \mathbb{R}$ and $u \in C(\Omega_T)$ satisfies $(\mathcal{P})$ with $u_0 > 0$ in Ω. If $u = 0$ in some interior point of Ω_T, then there exists $(x_1, t_1) \in \Omega_T$ and a ball $B_R(x_0, t_0) \subset \Omega_T$ with $(x_1, t_1) \in \partial B_R(x_0, t_0)$ such that*

$$u(x_1, t_1) = 0, \qquad u > 0 \quad \text{in } B_R(x_0, t_0), \quad \text{and} \quad t_1 > t_0.$$

Proof. Let $u(\bar{x}, \bar{t}\,) = 0$ for some $(\bar{x}, \bar{t}\,) \in \Omega_T$. Since $u_0 > 0$ in Ω and u is continuous in Ω_T, we may assume that there is no $t < \bar{t}$ such that $u(\bar{x}, t) = 0$. Moreover, there exists $R_1 > 0$ such that

$$u > 0 \quad \text{in } B_{R_1}(\bar{x}, 0) \cap \{t > 0\} \subset \Omega_T.$$

We move $B_{R_1}(\bar{x}, t)$ up by increasing $t > 0$, until there arises a point $(x,t) \in \Omega_T$ such that

$$u(x,t) = 0 \quad \text{and} \quad (x,t) \in \partial B_{R_1}(\bar{x}, \bar{t}_1). \tag{3.2}$$

Obviously, $t \geq \bar{t}_1$. If there exists a point (x_1, t_1) that satisfies (3.2) and $t_1 > \bar{t}_1$, then, taking a smaller ball which lies entirely in Ω_T, we get the desired result. Note also that $\bar{t}_1 + R_1 \leq \bar{t} < T < +\infty$.

Assume now that $t = \bar{t}_1$ for any (x,t) which satisfies (3.2). Taking some $R_2 < R_1$, we repeat the above construction and obtain $(x,t) \in \Omega_T$ and $B_{R_2}(\bar{x}, \bar{t}_2)$ such that

$$u(x,t) = 0 \quad \text{and} \quad (x,t) \in \partial B_{R_2}(\bar{x}, \bar{t}_2). \tag{3.3}$$

It is also clear that $t \geq \bar{t}_2 > \bar{t}_1$ and $\bar{t}_2 + R_2 \leq \bar{t} < +\infty$ (see Fig. 1). Again, either there exists (x_2, t_2) that satisfies (3.3) with $t_2 > \bar{t}_2$, or else $t = \bar{t}_2$ for any such (x,t) satisfying (3.3).

Let us show that there are some step $k \in \mathbb{N}$ and a point (x_k, t_k), such that $u(x_k, t_k) = 0$ and $(x_k, t_k) \in \partial B_{R_k}(\bar{x}, \bar{t}_k)$ with $t_k > \bar{t}_k$.

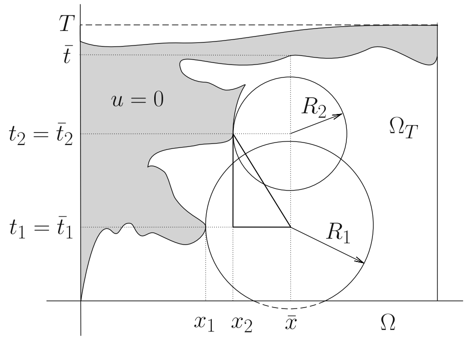

Fig. 1. Construction of the proof.

Suppose that, by contradiction, for every $k \in \mathbb{N}$ and for any $(x_k, t_k) \in \Omega_T$ such that $u(x_k, t_k) = 0$ and $(x_k, t_k) \in \partial B_{R_k}(\bar x, \bar t_k)$, we get $\bar t_k = t_k$. Note that $\bar t_k + R_k \le \bar t < +\infty$ for all $k \in \mathbb{N}$ and the sequence (t_k) is strictly increasing with

$$t_{k+1} > t_k + \sqrt{R_k^2 - R_{k+1}^2}. \tag{3.4}$$

Without lost of generality, we may assume $R_1 = 1$. Fix an integer $m \ge 2$ and take $R_{k+1} = 1 - \frac{k}{m}$ for $k = 1, 2, \dots, (m-1)$. Then, by (3.4), we get

$$\begin{aligned} t_m &> t_1 + \sum_{k=1}^{m-1} \sqrt{R_k^2 - R_{k+1}^2} = t_1 + \sum_{k=1}^{m-1} \sqrt{\frac{2}{m} - \frac{2k-1}{m^2}} \\ &\ge t_1 + \frac{1}{\sqrt{m}} \sum_{k=1}^{m-1} \sqrt{1 - \frac{k}{m}} \ge t_1 + \frac{1}{\sqrt{m}} \sum_{k=1}^{m-1} \left(1 - \frac{k}{m}\right) \\ &= t_1 + \frac{m-1}{2\sqrt{m}} \to +\infty \quad \text{as } m \to \infty. \end{aligned}$$

But this is impossible since $\bar t < T < +\infty$. This contradiction ends the proof. □

Now we are able to prove the SMP. The proof will be obtained by using techniques similar to [14, Theorem 2, p. 168].

Proof of Theorem 1.1. Beginning of the proof is the same for both cases, $1 < p < 2$ and $p > 2$.

Suppose the assertion of the theorem is false and u is zero in some interior point of Ω_T. Since $0 \not\equiv u_0 \ge 0$ and $u \ge 0$ on $\partial\Omega_T$ for all $p > 1$, we have $0 \not\equiv u \ge 0$ on Ω_T by the WMP that follows from Theorem 2.1. Therefore, there exist $(x_1, t_1) \in \Omega_T$ and a ball $B_R(x_0, t_0) \subset \Omega_T$ with $(x_1, t_1) \in \partial B_R(x_0, t_0)$, such that

$$u(x_1, t_1) = 0 \quad \text{and} \quad u > 0 \quad \text{in } B_R(x_0, t_0). \tag{3.5}$$

Taking a ball of smaller radius, if necessary, (x_1, t_1) becomes a unique zero point of u on the corresponding sphere $\partial B_R(x_0, t_0)$.

For some $B_r(x_1, t_1) \subset \Omega_T$ with $r \in (0, R)$ small enough we define the domain $D := B_R(x_0, t_0) \cap B_r(x_1, t_1)$ with the boundary $\partial D := C' \cup C''$, where (see Fig. 2)

$$C' := \partial B_R(x_0, t_0) \cap B_r(x_1, t_1), \qquad C'' := \overline{B}_R(x_0, t_0) \cap \partial B_r(x_1, t_1).$$

Note that by (3.5) we have $\varepsilon := \inf\{u(x, t)\colon (x, t) \in C''\} > 0$.

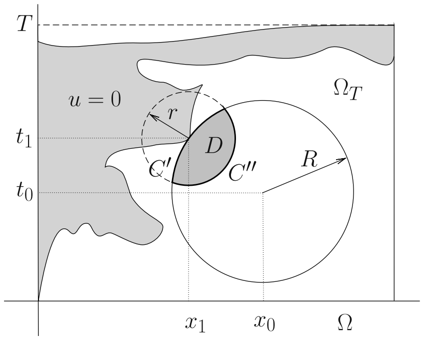

Fig. 2. Construction of the proof.

Consider now the function

$$v(x,t) := \varepsilon\bigl(e^{-\alpha d^2} - e^{-\alpha R^2}\bigr), \tag{3.6}$$

where $\alpha > 0$ and $d := d(x,t;x_0,t_0)$ is the Euclidean metric in $\mathbb{R}^{N+1}$ defined by (3.1). Evidently we have

$$0 < v \le \varepsilon \quad \text{in } B_R(x_0,t_0), \qquad v = 0 \quad \text{on } \partial B_R(x_0,t_0), \qquad v < 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R(x_0,t_0).$$

Moreover, by definition of ε we get $v \le u$ on ∂D for every $\alpha > 0$.

By direct calculations, for $(x,t) \in B_R(x_0,t_0)$ we obtain

$$\begin{aligned}\mathcal{L}_\lambda[v] = &-2\alpha\varepsilon e^{-\alpha d^2}(t-t_0) - (2\alpha)^{p-1}\varepsilon^{p-1}|x-x_0|^{p-2}e^{-(p-1)\alpha d^2}\bigl(2\alpha(p-1)|x-x_0|^2 - (p-2+N)\bigr)\\ &- \lambda\varepsilon^{p-1}e^{-(p-1)\alpha d^2}\bigl(1-e^{-\alpha(R^2-d^2)}\bigr)^{p-1} = -2\alpha\varepsilon e^{-\alpha d^2}H(x,t),\end{aligned}$$

where

$$\begin{aligned}H(x,t) := \frac{\varepsilon^{p-2}e^{-(p-2)\alpha d^2}}{2\alpha}&\bigl(\lambda\bigl(1-e^{-\alpha(R^2-d^2)}\bigr)^{p-1}\\ &+ (2\alpha)^{p-1}|x-x_0|^{p-2}\bigl(2\alpha(p-1)|x-x_0|^2 - (p-2+N)\bigr)\bigr) + (t-t_0).\end{aligned} \tag{3.7}$$

The rest of the proof will be different for $p > 2$ and $1 < p < 2$.

1. Let $p > 2$ and $u_0 > 0$. By Lemma 3.1 we may choose the centers $(x_0,t_0), (x_1,t_1) \in \Omega_T$ and radii $R, r > 0$ of the balls $B_R(x_0,t_0)$, $B_r(x_1,t_1)$ such that $t_1 - r > t_0$. Since for $p > 2$ every term in (3.7) is bounded, we choose $\alpha > 0$ sufficiently large such that $H(x,t) \ge 0$ in D, and consequently $\mathcal{L}_\lambda[v] \le 0$ in this domain. Hence, $\mathcal{L}_\lambda[v] \le \mathcal{L}_\lambda[u]$. Using the fact that $v \le u$ on ∂D, we apply the WCP from Lemma 2.3 and derive that $v \le u$ in D. Thus, since $u \ge 0$ in Ω_T and $u \in C^1(\Omega_T)$, we get

$$\begin{aligned}0 = \nabla u(x_1,t_1)\cdot(t_1-t_0, x_1-x_0) = \frac{\partial u(x_1,t_1)}{\partial\nu} \le \frac{\partial v(x_1,t_1)}{\partial\nu}\\ = \nabla v(x_1,t_1)\cdot(t_1-t_0,x_1-x_0) = -2\alpha\varepsilon R^2 e^{-\alpha R^2} < 0,\end{aligned}$$

where ν is the outer unit normal to $B_R(x_0,t_0)$. Consequently, we have a contradiction and $u > 0$ in Ω_T.

2. Let $1 < p < 2$. If $x_0 \ne x_1$ then we are able to choose the radius $r \in (0,R)$ so small that $|x-x_0| > r$ holds for all $(x,t) \in \overline{D}$. In this case, since $1 < p < 2$, we may choose a sufficiently large $\alpha > 0$ such that $H(x,t) \ge 0$ in D. Therefore, $\mathcal{L}_\lambda[v] \le 0 \le \mathcal{L}_\lambda[u]$ and we get a contradiction, arguing as in the case $p > 2$.

Assume now that $x_0 = x_1$. Applying a direct $(N+1)$-dimensional generalization of Lemma 2 from [14, p. 166], we obtain $u(\cdot,t_1) \equiv 0$ in Ω. Consider

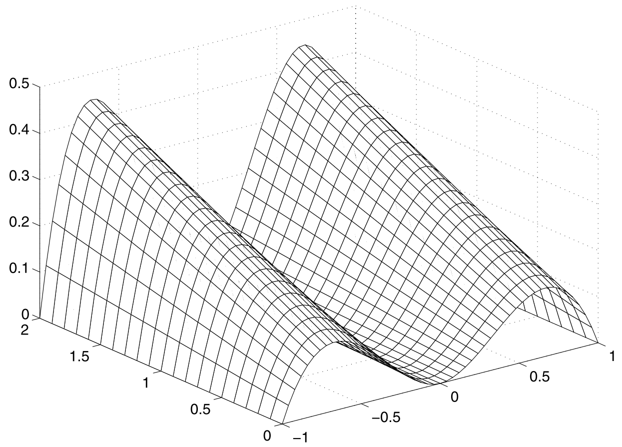

Fig. 3. The graph of $v(x,t)$ for $k=1$.

$$\bar{t} := \inf\bigl\{t\colon\ \exists x \in \Omega \text{ such that } u(x,t)=0\bigr\}. \tag{3.8}$$

If $\bar{t}=0$, then we get a contradiction to $u_0 \not\equiv 0$ in Ω. Thus, $\bar{t}>0$ and consequently $u>0$ in $\Omega_{\bar{t}}$. □

Proof of Theorem 1.2. The proof follows immediately from the proof of Part 2 of Theorem 1.1. □

Now we prove that in the case $1<p<2$, an arbitrary supersolution of $(\mathcal{P})$ has a negative outer normal derivative at every point on $\partial\Omega_T$ at which it is equal to zero.

Proof of Theorem 1.3. The proof is identical to the proof of Theorem 1.1; it is based on an application of the WCP to $u(x,t)$ and the test function $v(x,t)$ given by formula (3.6). □

Remark 3.2. We are not able to prove the HMP for the case $p>2$ using the techniques of Theorem 1.1, since it works only for $t_1>t_0$. But it is impossible to prove the HMP for $p>2$, in general, see Counterexample 3 in Section 4.

4. Counterexamples

In this section we show some counterexamples to Theorems 1.1 and 1.3 if certain assumptions of these theorems are violated.

Counterexample 1. In the case $p>2$ we cannot discard the assumption $u_0>0$ in Ω and prove the SMP only under the assumptions $u_0 \ge 0$ and $u_0 \not\equiv 0$ in Ω.

Indeed, consider the function

$$v(x,t) := (t+1)x^{2k}\bigl(1-|x|\bigr), \tag{4.1}$$

for $x\in(-1,1)$, $t\in(0,T)$ for any $T<\infty$ and $k\in\mathbb{N}$ (see Fig. 3). Let us show that for any $\lambda\in\mathbb{R}$ and $p>2$ there exist appropriate $\varepsilon>0$ and $k\in\mathbb{N}$ such that $\mathcal{L}_\lambda[\varepsilon v]\ge 0$. By direct calculations we get

$$\begin{aligned}\mathcal{L}_\lambda[\varepsilon v] &= \varepsilon x^{2k}\bigl(1-|x|\bigr) + \varepsilon^{p-1}(p-1)(t+1)^{p-1}2k|x|^{(2k-1)(p-2)+(2k-2)} \\ &\quad \times \bigl|2k-(2k+1)|x|\bigr|^{p-2}\bigl((2k+1)|x|-(2k-1)\bigr) - \varepsilon^{p-1}\lambda(t+1)^{p-1}x^{2k(p-1)}\bigl(1-|x|\bigr)^{p-1}. \end{aligned} \tag{4.2}$$

Since $p > 2$, taking $\varepsilon > 0$ sufficiently small, we make the first summand strictly greater than the third one for all x such that

$$1 > |x| \geq \frac{2k-1}{2k+1} = 1 - \frac{2}{2k+1}. \tag{4.3}$$

Moreover, the second summand becomes nonnegative on this interval. Hence, we obtain $\mathcal{L}_\lambda[\varepsilon v] \geq 0$ for all x that satisfy (4.3).

On the other hand, if the power of $|x|$ satisfies

$$(2k-1)(p-2) + (2k-2) \geq 2k, \tag{4.4}$$

then there exists $\varepsilon > 0$ small enough such that $\mathcal{L}_\lambda[\varepsilon v] \geq 0$ for all $|x| < 1 - 2/(2k+1)$. From (4.4) we get

$$p \geq 2 + \frac{2}{2k-1} \to 2 \quad \text{as } k \to +\infty.$$

Thus, for every $p > 2$ there exists $k \in \mathbb{N}$ with property (4.4). Consequently, we are able to chose $\varepsilon > 0$ sufficiently small, such that $\mathcal{L}_\lambda[\varepsilon v] \geq 0$ for $x \in (-1, 1)$ and $t \in (0, T)$. Hence, we get a contradiction to the SMP, since $u(0, t) = 0$ for any $t \in (0, T)$.

Remark 4.1. Since every term in (4.2) is bounded, we get $\mathcal{L}_\lambda[\varepsilon v] \in L^\infty([-1, 1] \times [0, T])$ for every $\varepsilon > 0$ and every $T < \infty$.

Remark 4.2. Notice that for $p > 2$, and assuming only $u_0 \geq 0$ and $u_0 \not\equiv 0$ in Ω, we get a contradiction also to the *Strong Comparison Principle*. This claim is easily derived from Eq. (4.2) by considering

$$w_1(x,t) = \varepsilon_1 v(x,t) \quad \text{and} \quad w_2(x,t) = \varepsilon_2 v(x,t),$$

with $v(x,t)$ defined by (4.1) and $\varepsilon_1, \varepsilon_2 > 0$. We are able to find ε_1, ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$ and

$$\mathcal{L}_\lambda[w_1] \leq \mathcal{L}_\lambda[w_2], \qquad w_1(x,0) \leq w_2(x,0).$$

But, at the same time, $w_1(0,t) = w_2(0,t)$ for all $t \in [0, T]$.

Remark 4.3. The counterexample given by formula (4.1) can be generalized to the $(N+1)$-dimensional case by considering the radial version of (4.1). Another counterexample to the SMP for $p > 2$ is constructed in [11, p. 97].

Counterexample 2. In the case $1 < p < 2$, the "local" (in time) SMP that holds throughout some cylinder $\Omega_{\bar{t}}$, $0 < \bar{t} < T$, cannot be extended to the SMP on the whole space–time domain Ω_T.

Consider the function

$$v(x,t) := (t-1)^{2k} \sin_p(x), \tag{4.5}$$

for $t \geq 0$, $x \in (0, \pi_p)$ and $k \in \mathbb{N}$ (see Fig. 4). Here $\sin_p(x)$ is a generalization of the basic sine function (see Lindqvist [9]), and it is the first eigenfunction of the operator $-\Delta_p$ on $(0, \pi_p)$ with zero boundary conditions, where π_p is the first positive zero of $\sin_p(x)$; i.e.,

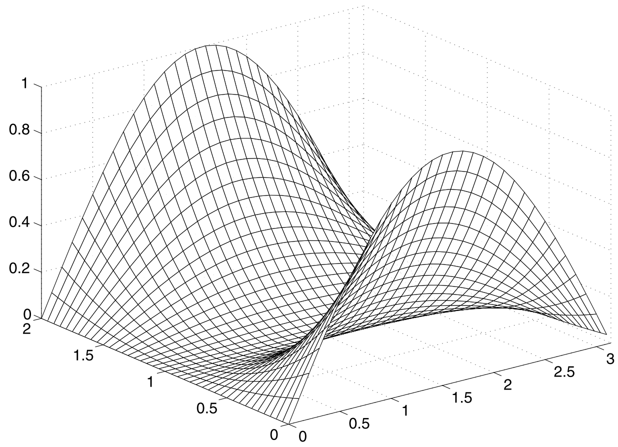

Fig. 4. The graph of $v(x,t)$ for $k=1$.

$$\left.\begin{aligned} &-\left(\left|u'\right|^{p-2}u'\right)' = \lambda_1|u|^{p-2}u, \qquad && x\in(0,\pi_p);\\ &u(0)=u(\pi_p)=0; && \\ &u>0, && x\in(0,\pi_p), \end{aligned}\right\} \tag{4.6}$$

where λ_1 is the corresponding first eigenvalue.

Obviously, $v(x,0)=\sin_p(x)\ge 0$ on $[0,\pi_p]$ and $v(0,t)=v(\pi_p,t)=0$ for all $t\ge 0$. Moreover, $v(x,1)\equiv 0$ on $[0,\pi_p]$.

Let us show that, for any $\lambda<\lambda_1$, there exists an appropriate $\varepsilon>0$ such that $\mathcal{L}_\lambda[\varepsilon v]\ge 0$. Using (4.6), by simple calculations we get

$$\mathcal{L}_\lambda[\varepsilon v] = \varepsilon \sin_p(x)\left(2k(t-1)^{2k-1} + \varepsilon^{p-2}(\lambda_1-\lambda)\left|\sin_p(x)\right|^{p-2}(t-1)^{2k(p-1)}\right). \tag{4.7}$$

Since $\sin_p(x)\ge 0$ on $[0,\pi]$, we only need to find $\varepsilon>0$ such that

$$2k(t-1)^{2k-1} + \varepsilon^{p-2}(\lambda_1-\lambda)\left|\sin_p(x)\right|^{p-2}(t-1)^{2k(p-1)} \ge 0.$$

The case $t\ge 1$ immediately yields $\mathcal{L}_\lambda[\varepsilon v]\ge 0$ for all $\varepsilon>0$ and $\lambda\le\lambda_1$. Let now $0\le t<1$. Since $|\sin_p(x)|^{p-2}\ge 1$ on $(0,\pi_p)$, we need

$$\varepsilon^{p-2}(\lambda_1-\lambda)(1-t)^{2k(p-1)} \ge 2k(1-t)^{2k-1}.$$

Hence, if $\lambda<\lambda_1$ and the power $2k-1-2k(p-1)\ge 0$, then there exists $\varepsilon>0$, which doesn't depend on t, such that $\mathcal{L}_\lambda[\varepsilon v]\ge 0$. This is possible if and only if

$$p\le 2-\frac{1}{2k}.$$

Thus, for any $p\in(1,2)$ we are able to choose $k\in\mathbb{N}$ such that this inequality holds.

As noted above $v(x,1)\equiv 0$ on $[0,\pi_p]$ but $v(x,t)>0$ in another cases; i.e., the Strong Maximum Principle doesn't hold for $1<p<2$ on Ω_T for any $T>1$.

Remark 4.4. From (4.7) it can be easily deduced that $\mathcal{L}_\lambda[\varepsilon v]\in L^\infty([0,\pi_p]\times[0,T])$ for all $T>0$.

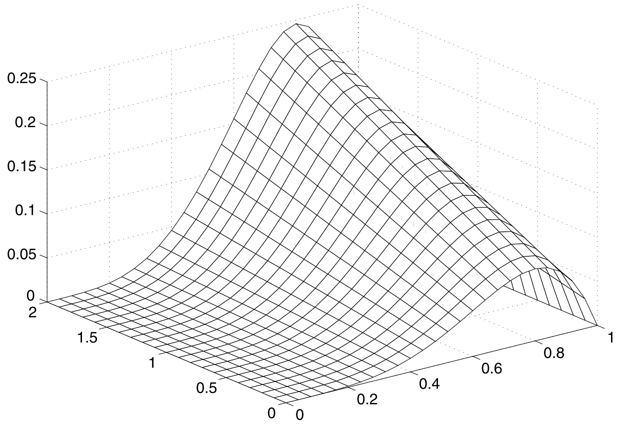

Fig. 5. The graph of $v(x,t)$ for $k=2$.

Remark 4.5. It is easy to see that the SCP also doesn't hold. Indeed, considering

$$w_1(x,t)=\varepsilon_1 v(x,t) \quad \text{and} \quad w_2(x,t)=\varepsilon_2 v(x,t),$$

with appropriate $k\in\mathbb{N}$ and $\varepsilon_1,\varepsilon_2>0$ such that $\varepsilon_2>\varepsilon_1$, we get a contradiction to the SCP.

Remark 4.6. The counterexample given by formula (4.5) can be generalized to the $(N+1)$-dimensional case by using the same technique with the first eigenpair (φ_1,λ_1) of $-\Delta_p$ on $\Omega\subset\mathbb{R}^N$ with zero Dirichlet boundary conditions.

Even a "stronger" counterexample to the SMP in Ω_T than our Counterexample 2 is constructed in E. DiBenedetto, U.P. Gianazza, and V. Vespri [7, pp. 64–65], Chapter 4, §3.3: *For some* $\bar{t}\in(0,T)$, *one has* $u>0$ *in* $\Omega_{\bar{t}}$, *whereas* $u=0$ *in* $\Omega_T\setminus\Omega_{\bar{t}}$.

Counterexample 3. We show that the Hopf Maximum Principe for $p\in(12)$, given by Theorem 1.3, cannot be generalized to the case $p>2$.

Consider the function from Counterexample 1:

$$v(x,t):=(t+1)x^{2k}(1-x), \tag{4.8}$$

for $x\in(0,1)$ only, $t\in(0,T)$ for any $T<\infty$ and $k\in\mathbb{N}$ (see Fig. 5). Using similar arguments, we conclude that for every $p>2$ there exists $k\in\mathbb{N}$ such that $\mathcal{L}_\lambda[\varepsilon v]\geq 0$ for $x\in(0,1)$ and $t\in(0,T)$. Since for $x=0$ and $t\in[0,T]$ we have

$$\frac{\partial u(x,t)}{\partial\nu}=-\nabla_x u(x,t)=0,$$

we get a contradiction to the HMP.

Remark 4.7. It is also possible to construct an $(N+1)$-dimensional counterexample to the HMP for $p>2$, by considering the radial version of the following function

$$v(x,t):=(t+1)\big(1+2|x|\big)\big(1-|x|\big)^2$$

for $x\in(-1,1)$ and $t\in(0,T)$, $T<\infty$.

References

[1] A. Anane, Simplicité et isolation de la premiere valeur propre du p-laplacien avec poids, C. R. Acad. Sci., Sér. 1 305 (16) (1987) 725–728.
[2] M. Cuesta, P. Takáč, A strong comparison principle for the Dirichlet p-Laplacian, in: G. Caristi, E. Mitidieri (Eds.), Proceedings of the Conference on Reaction–Diffusion Equations, Trieste, Italy, October 1995, in: Lect. Notes Pure Appl. Math., vol. 194, Marcel Dekker, Inc., New York–Basel, 1998, pp. 79–87.
[3] M. Cuesta, P. Takáč, A strong comparison principle for positive solutions of degenerate elliptic equations, Differential Integral Equations 13 (4–6) (2000) 721–746.
[4] E. DiBenedetto, Degenerate Parabolic Equations, Springer, 1993.
[5] E. DiBenedetto, U.P. Gianazza, V. Vespri, Subpotential lower bounds for nonnegative solutions to certain quasi-linear degenerate parabolic equations, Duke Math. J. 143 (1) (2008) 1–15.
[6] E. DiBenedetto, U.P. Gianazza, V. Vespri, Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2) (2010) 385–422.
[7] E. DiBenedetto, U.P. Gianazza, V. Vespri, Harnack's Inequality for Degenerate and Singular Parabolic Equations, Springer, 2012.
[8] A. Friedman, Partial Differential Equations, RE Krieger Publishing Company, 1976.
[9] P. Lindqvist, Some remarkable sine and cosine functions, Ric. Mat. XLIV (1995) 269–290.
[10] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[11] B. Nazaret, Principe du maximum strict pour un opérateur quasi linéaire, C. R. Acad. Sci., Ser. 1 Math. 333 (2) (2001) 97–102.
[12] J. Nečas, Les Méthodes Directes en Théorie des Equations Elliptiques, Masson, Paris, 1967.
[13] J.F. Padial, P. Takáč, L. Tello, An antimaximum principle for a degenerate parabolic problem, Adv. Differential Equations 15 (7–8) (2010) 601–648.
[14] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, Prentice–Hall, Englewood Cliffs, NJ, 1967.
[15] P. Pucci, J. Serrin, The Maximum Principle, vol. 73, Birkhäuser, Basel, 2007.
[16] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1) (1984) 191–202.
[17] J. Vétois, Strong maximum principles for anisotropic elliptic and parabolic equations, Adv. Nonlinear Stud. 12 (1) (2012) 101–114.